Division of the two-qubit Hilbert space according to the entanglement sudden death under composite noise environment

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We show theoretically that according to the disentanglement behavior under composite noise environment, the Hilbert space of a two-qubit system can be divided into two separate parts: a 3-dimensional subspace in which all states disentangle asymptotically, and the rest in which all states disentangle abruptly. The violation of additivity for entanglement decay rates under weak noises [see, PRL **97**, 140403 (2006)] therefore can be explained in terms of such division of the Hilbert space.

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I. INTRODUCTION

Quantum coherence, as a consequence of the superposition principle, lies in the heart of quantum mechanics and is regarded as a unique sign for the quantum regime. A fundamental question raised by quantum coherence is the the emergence of macroscopic classicality from the microscopic quantum world. It is currently well-recognized that the answer lies in the so-called *decoherence* process, a dynamical evolution of the system in which quantum coherence is gradually lost due to the ubiquitous system-environment interaction [1]. Decoherence has been extensively studied for years in the realm of quantum optics [2] as a decay of *local* coherence. However, it was not until recently that the evolution of *nonlocal* quantum coherence, i.e., the *entanglement dynamics*, has become a field of interest that attracts growing theoretical and experimental studies [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Moreover, since entanglement itself is a key resource for quantum computation and information processing [18, 19], such studies not only put new insights to the fundamentals of quantum mechanics, but also scrutinize more stringently on whether and how one can build an applicable quantum computer under realistic circumstances.

A remarkable finding in the study of entanglement dynamics is the entanglement sudden death (ESD), which was previously shown in [3] and latterly coined by Yu and Eberly [4, 5, 6]. The model studied in [4] consists of two initially entangled but non-interacting qubits each coupled to its own local environment. In all circumstances the entanglement between the two qubits disappears subsequently due to the spontaneous decay of each qubit. Very interestingly, the entanglement dynamics falls into two distinct categories: abrupt and asymptotic disentanglement, depending on the initial states. The abrupt disentanglement is characterized by a complete disappearance of entanglement within finite time and is thus termed ESD. The significance of ESD lies in two-folds: (i) It unveils a fundamental difference between local and nonlocal quantum coherences via their evolution dynamics; (ii) It puts an upper bound on the applicability of entangled pairs in practical quantum communication even with the best protocol for entanglement distillation.

Besides the apparent dependence on initial states, entanglement dynamics is also intrinsically impacted by the specific form of the system-environment interaction. Yu and Eberly also demonstrated examples of ESD under composite noise environment [5] and classical noise environment [6]. Structured reservoirs with memory effects add a new ingredient, *entanglement revival*, to the entanglement dynamics, due to the non-Markovian nature of the system-reservoir interation [7]. Ikram *et al* studied the effects of squeezed and thermal reservior [8]. Al-Qasimi *et al* [9] showed that all X-states undergo ESD when the reservoir is at finite temperature. Extensions of Yu and Eberly's model to commonly shared environment (Dicke-regime) [10] and high dimensional bipartite systems (e.g., the $3 \otimes 3$ system [11] and the continuous variable system [12]) have also been studied recently. Moreover, the question on where the lost entanglement goes has been addressed recently [13, 14], revealing a rich and counterintuitive relation between the ESD and the entanglement sudden birth in the reservoirs.

However, in most studies [4, 5, 6, 7, 8, 9] the two-qubit state under investigation is artificially confined to certain simple classes (e.g., Bell-like states, Werner-like states, or the X-states) for the convenience of theoretical handling. This led to a lack of global knowledge on how the full 4-dimensional two-qubit Hilbert space is structured according to the two distinct behaviors of entanglement decay. From the perspective of quantum communication, it is very important to know which state is more robust than others in fighting against entanglement lost. Thus it is desirable to discriminate states with asymptotic disentanglement behavior from others. Very recently, Huang and Zhu [15] have specified the necessary and sufficient condition for ESD under amplitude damping and phase

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damping, respectively, using a principal-minors technique based on the positive partial transpose (PPT) criterion. The purpose of this paper is to show that, even under the combined action of these two noises, there exists a 3-dimensional ESD-free subspace, in which all state disentangles asymptotically. Furthermore, we have proved that all states outside this subspace disentangle abruptly under the composite noise environment. Thus the full Hilbert space is completely divided into two separate parts determined solely upon whether they undergo ESD or not. Such a division of the Hilbert space also provides a simple explanation to the violation of additivity for entanglement decay rates under weak noises [5] via the restriction of the ESD-free subspace on the system dynamics.

This paper is organized as follows: In Section II we prove that the combined action of amplitude damping and phase damping can be factored as subsequent actions of each individually. Therefore the set of ESD-free states under composite noise environment is contained in the intersection of the two sets of ESD-free states under each of the constituent noises. In Section III we show that under pure phase damping, the ESD-free states constitute four 3-dimensional subspaces. In Section IV we prove that only one of the four subspaces is ESD-free under the composite noise environment. Finally, we make a few remarks and give a short conclusion in Section V.

II. THE FACTORIZATION OF AMPLITUDE DAMPING AND PHASE DAMPING

The system we study includes two non-interacting qubits (A and B) and two independent local environments, which will be modeled as either amplitude damping, or phase damping, or the combined action of both.

The master-equation of the system under amplitude damping and phase damping can be written as

$$\dot{\rho} = \sum_{A,B} \frac{\Gamma_1^{A,B}}{2}(2\sigma_-^{A,B}\rho\sigma_+^{A,B} - \sigma_+^{A,B}\sigma_-^{A,B}\rho - \rho\sigma_+^{A,B}\sigma_-^{A,B}) \tag{1}$$

and

$$\dot{\rho} = \sum_{A,B} \frac{\Gamma_2^{A,B}}{2}(\sigma_z^{A,B}\rho\sigma_z^{A,B} - \rho), \tag{2}$$

respectively. The two-qubit density operator ρ is represented in the computational basis $|1\rangle = |\uparrow\uparrow\rangle$, $|2\rangle = |\uparrow\downarrow\rangle$, $|3\rangle = |\downarrow\uparrow\rangle$, $|4\rangle = |\downarrow\downarrow\rangle$. The parameters $\Gamma_1^{A,B}$ and $\Gamma_2^{A,B}$ (for party A, B) are the population relaxation rate and dephasing rate for amplitude damping and phase damping, respectively. The operators $\sigma_+^{A,B}$, $\sigma_-^{A,B}$ and $\sigma_z^{A,B}$ denote the raising, lowering and Pauli operators for each qubit. The explicit time evolution of Eq.(1) and (2) can be solved and expressed in a unified form by the Kraus-operators as

$$\rho(t) = \sum_{i=1,2,3,4} K_i \rho(0) K_i^\dagger, \tag{3}$$

with $\sum_i K_i^\dagger K_i = 1$. The two-qubit Kraus-operator K_i's can be expressed as tensor product of the one-qubit Kraus-operators for each party. We specifically denote K_i^{AM} as the two-qubit Kraus-operator under amplitude damping defined by $K_1^{Am} = M_1^A \otimes M_1^B$, $K_2^{Am} = M_1^A \otimes M_2^B$, $K_3^{Am} = M_2^A \otimes M_1^B$, $K_4^{Am} = M_2^A \otimes M_2^B$, with

$$M_1^{A,B} = \begin{bmatrix} \gamma_1^{A,B} & 0 \\ 0 & 1 \end{bmatrix}, M_2^{A,B} = \begin{bmatrix} 0 & 0 \\ \omega_1^{A,B} & 0 \end{bmatrix}, \tag{4}$$

where $\gamma_1^{A,B} = \exp(-\frac{1}{2}\Gamma_1^{A,B}t)$ and $\omega_1^{A,B} = \sqrt{1-(\gamma_1^{A,B})^2}$. While the Kraus-operator K_i^{Ph} for phase damping is defined by $K_1^{Ph} = P_1^A \otimes P_1^B$, $K_2^{Ph} = P_1^A \otimes P_2^B$, $K_3^{Ph} = P_2^A \otimes P_1^B$, $K_4^{Ph} = P_2^A \otimes P_2^B$, with

$$P_1^{A,B} = \begin{bmatrix} \gamma_2^{A,B} & 0 \\ 0 & 1 \end{bmatrix}, P_2^{A,B} = \begin{bmatrix} \omega_2^{A,B} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where $\gamma_2^{A,B} = \exp(-\Gamma_2^{A,B}t)$ and $\omega_2^{A,B} = \sqrt{1-(\gamma_2^{A,B})^2}$. When both noises participate, the master-equation reads

$$\begin{aligned} \dot{\rho} &= \sum_{A,B} \frac{\Gamma_1^{A,B}}{2}(2\sigma_-^{A,B}\rho\sigma_+^{A,B} - \sigma_+^{A,B}\sigma_-^{A,B}\rho - \rho\sigma_+^{A,B}\sigma_-^{A,B}) \\ &+ \sum_{A,B} \frac{\Gamma_2^{A,B}}{2}(\sigma_z^{A,B}\rho\sigma_z^{A,B} - \rho). \end{aligned} \tag{6}$$

Similarly, the time evolution of Eq.(6) can be expressed by the Kraus-operators as

$$\rho(t) = \sum_{k,l=1,2,3} \left(C_k^A \otimes C_l^B\right) \rho(0) \left(C_k^A \otimes C_l^B\right)^\dagger, \tag{7}$$

where

$$\begin{aligned} C_1^{A,B} &= \begin{bmatrix} \gamma_1^{A,B}\gamma_2^{A,B} & 0 \\ 0 & 1 \end{bmatrix}, \\ C_2^{A,B} &= \begin{bmatrix} \gamma_1^{A,B}\omega_2^{A,B} & 0 \\ 0 & 0 \end{bmatrix}, \\ C_3^{A,B} &= \begin{bmatrix} 0 & 0 \\ \omega_1^{A,B} & 0 \end{bmatrix}. \end{aligned} \tag{8}$$

Using the above Kraus-operators it is straightforward to verify the following equations

$$\begin{aligned} &\sum_{k,l=1,2,3} \left(C_k^A \otimes C_l^B\right) \rho(0) \left(C_k^A \otimes C_l^B\right)^\dagger \\ &= \sum_{j=1,2,3,4} K_j^{Am} \left[\sum_{i=1,2,3,4} K_i^{Ph} \rho(0) K_i^{Ph\dagger}\right] K_j^{Am\dagger} \\ &= \sum_{i=1,2,3,4} K_i^{Ph} \left[\sum_{j=1,2,3,4} K_j^{Am} \rho(0) K_j^{Am\dagger}\right] K_i^{Ph\dagger}, \end{aligned} \tag{9}$$

which can be further packaged into a compact form as

$$\$_t^C = \$_t^{Am}\$_t^{Ph} = \$_t^{Ph}\$_t^{Am}, \tag{10}$$

where $\$_t^{Am}$, $\$_t^{Ph}$, and $\$_t^C$ denote the quantum channels governing the evolution of the system under amplitude damping, phase damping, and both of them, respectively. The essence of Eq.(10) is that the effect of the composite noise environment can be factored as subsequent actions of each constituent noise environment. Moreover, as a consequence of the Markovian master-equations, it is also straightforward to verify a general time-domain factorization property bellow

$$\$_t = \$_{\tau'}\$_\tau, \tag{11}$$

where $t = \tau + \tau'$, $\tau, \tau' \geq 0$. Eq.(11) holds for every kind of the quantum channels $\$_t^{Ph}$, $\$_t^{Am}$ and $\$_t^C$. Since the entanglement is non-increasing under local operations (note that all the quoted quantum channels are local operations) [18, 19], a direct conclusion drawn from the factorization properties Eq.(10) and (11) is that if a state has completely disentangled under either $\$_t^{Am}$ or $\$_t^{Ph}$, it must remain disentangled under the action of $\$_t^C$. In other words, the set of the ESD-free states under the composite noise environment is a subset of the intersection of the sets of ESD-free states under each one of the noises. Therefore, to be ESD-free under the action of each noise is a necessary condition to be ESD-free under the action of both noises.

Finally, we would like to clarify that amplitude damping itself does not merely produce damping in the excited state of the qubit, but also causes dephasing between the two states. This concomitant dephasing has been automatically included in the the master-equation (1) and the quantum channel $\$_t^{Am}$. In other words, it is impossible to factor $\$_t^{Am}$ further into a fictitious "pure amplitude damping" channel and a fictitious "pure dephasing" channel. Therefore, when we talk about the composite noise environment we mean that an extra dephasing noise, besides the dephasing caused by amplitude damping, is present in the system-reservoir interaction. Such composite noise environment is a reasonable model for realistic circumstances where the noisy environment can be eventually treated as a mixture of the two noises.

III. THE ESD-FREE SUBSPACE UNDER PHASE DAMPING

We have shown that to find the ESD-free states under the composite noise environment, it is sufficient to explore within the set of the ESD-free states under either one of the two noises. Our strategy is to investigate the set of ESD-free states under phase damping first, since there is a simple partition between the ESD and the ESD-free states for pure phase noise. The question for the composite noise environment will be addressed in the next section.

The necessary and sufficient condition for ESD under phase damping has been specified by Huang and Zhu [15] very recently using the PPT criterion. The method they use is technically tedious and lack of physical transparency. For the completeness of narration, we re-derive it using a more intelligible method based on the concurrence [20]. The concurrence $C(\rho)$ is an entanglement monotone ranges from 0 to 1 and can be computed directly from the two-qubit density matrix ρ as

$$C(\rho) = \max(0, \Lambda), \tag{12}$$

where $\Lambda = \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}$, with the λ_i's the eigenvalues of the matrix

$$R(\rho) = \rho\,(\sigma_y \otimes \sigma_y)\,\rho^*\,(\sigma_y \otimes \sigma_y) \tag{13}$$

in descending order.

We first consider the concurrence under phase damping in the infinite-time limit, i.e., under the action of $\$_\infty^{Ph}$. Suppose the initial state is

$$\rho(0) = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{bmatrix}. \tag{14}$$

After infinite time of phase damping it is found

$$\rho(\infty) = \$_\infty^{Ph}\rho(0) = \begin{bmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{bmatrix}. \tag{15}$$

The concurrence of the above final state is simply

$$C[\rho(\infty)] = \max\left[0, \Lambda(\infty)\right], \tag{16}$$

where

$$\Lambda(\infty) = \begin{cases} -2\sqrt{\rho_{11}\rho_{44}}, & \rho_{22}\rho_{33} \geq \rho_{11}\rho_{44}; \\ -2\sqrt{\rho_{22}\rho_{33}}, & \rho_{22}\rho_{33} < \rho_{11}\rho_{44}. \end{cases} \tag{17}$$

Suppose $\rho_{11}\rho_{22}\rho_{33}\rho_{44} \neq 0$, it is obvious that

$$\Lambda(\infty) < 0. \tag{18}$$

Since $\Lambda(t)$ is an algebraic function of the eigenvalues of $R[\rho(t)]$, it must be a continuous function of t. The inequality (18) thus means that if $\Lambda(t)$ starts from a positive value, it must have crossed over zero before $t = \infty$, i.e., the two-qubit system must have disentangled at a finite time. That is to say, the necessary condition for a state to be ESD-free under phase damping is to have *at least one vanishing diagonal element* in its density matrix. Thus any ESD-free state must belong to one of the four subspaces below

$$\rho_I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & \rho_{24} \\ 0 & \rho_{32} & \rho_{33} & \rho_{34} \\ 0 & \rho_{42} & \rho_{43} & \rho_{44} \end{bmatrix},$$

$$\rho_{II} = \begin{bmatrix} \rho_{11} & 0 & \rho_{13} & \rho_{14} \\ 0 & 0 & 0 & 0 \\ \rho_{31} & 0 & \rho_{33} & \rho_{34} \\ \rho_{41} & 0 & \rho_{43} & \rho_{44} \end{bmatrix},$$

$$\rho_{III} = \begin{bmatrix} \rho_{11} & \rho_{12} & 0 & \rho_{14} \\ \rho_{21} & \rho_{22} & 0 & \rho_{24} \\ 0 & 0 & 0 & 0 \\ \rho_{41} & \rho_{42} & 0 & \rho_{44} \end{bmatrix},$$

$$\rho_{IV} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & 0 \\ \rho_{21} & \rho_{22} & \rho_{23} & 0 \\ \rho_{31} & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

To prove that all states in $\rho_{I\sim IV}$ are indeed ESD-free under phase damping, we further investigate the concurrence at finite time t, e.g., for an initial state $\rho_I(0)$ located in the subspace ρ_I. We find the eigenvalues of $R[\rho_I(t)] = R[\$_t^{Ph}\rho_I(0)]$ are $\lambda_{1,2} = \alpha \pm \beta$, $\lambda_{3,4} = 0$, where $\alpha = (\gamma_2^A\gamma_2^B)^2\rho_{23}\rho_{32} + \rho_{22}\rho_{33}$ and $\beta = 2\gamma_2^A\gamma_2^B\sqrt{\rho_{23}\rho_{32}\rho_{22}\rho_{33}}$. Thus

$$\begin{aligned} \Lambda(t) &= \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \\ &= 2\beta/(\sqrt{\alpha+\beta} + \sqrt{\alpha-\beta}). \end{aligned} \tag{20}$$

It is clear that $\Lambda(t) \geq 0$ holds $\forall\, t$. Only in two cases $\Lambda(t) = 0$: (i) $t = \infty$, which means asymptotic disentanglement; (ii) $\rho_{23} = \rho_{32} = 0$, which means $\Lambda(t) = 0$ holds $\forall\, t$, i.e., the state $\rho_I(0)$ is seperable from beginning. Thus all states in ρ_I are indeed ESD-free under phase damping. Straightforward calculations show that the same conclusion holds for the other three subspaces ρ_{II}, ρ_{III} and ρ_{IV}. Thus, a generic state $\rho(0)$ is ESD-free under phase damping iff

$$\rho_{11}\rho_{22}\rho_{33}\rho_{44} = 0, \tag{21}$$

or equivalent to say, iff it is located within the four subspaces $\rho_{I\sim IV}$ shown in Eq.(19).

IV. THE ESD-FREE SUBSPACE UNDER COMPOSITE NOISE ENVIRONMENT

Now we further explore the entanglement dynamics for states in the subspace $\rho_{I\sim IV}$ under the combined action of phase damping and amplitude damping. We find completely different disentanglement behaviors between the subspace ρ_I and the other three subspaces $\rho_{II\sim IV}$: all states in ρ_I disentangle asymptotically and all states in ρ_{II}, ρ_{III} and ρ_{IV} disentangle abruptly.

A. All states in ρ_I disentangle asymptotically

It is straightforward to show that the subspace ρ_I is full ESD-free whatever the noisy environment is. For amplitude damping, the evolution of an arbitrary initial state $\rho_I(0)$ is expressed as $\rho_I'(t) = \$_t^{Am}\rho_I(0)$ and the eigenvalues of $R[\rho_I'(t)]$ are $\lambda_{1,2}' = \alpha' \pm \beta'$ and $\lambda_{3,4}' = 0$, where $\alpha' = (\gamma_1^A\gamma_1^B)^2(\rho_{23}\rho_{32} + \rho_{22}\rho_{33})$ and $\beta' = 2(\gamma_1^A\gamma_1^B)^2\sqrt{\rho_{23}\rho_{32}\rho_{22}\rho_{33}}$. Obviously, $\beta' \geq 0$ and the equality holds only for $t = \infty$ or $\rho_{23} = \rho_{32} = 0$. Following the same reasoning as for Eq.(20), it is clear that the subspace ρ_I is also ESD-free under amplitude damping. Similarly, when both noises are present, the evoluation of $\rho_I(0)$ is simply $\rho_I''(t) = \$_t^C\rho_I(0)$ and the eigenvalues of $R[\rho_I''(t)]$ are $\lambda_{1,2}'' = \alpha'' \pm \beta''$ and $\lambda_{3,4}'' = 0$, where $\alpha'' = (\gamma_1^A\gamma_1^B)^2[(\gamma_2^A\gamma_2^B)^2\rho_{23}\rho_{32} + \rho_{22}\rho_{33}]$ and $\beta'' = 2(\gamma_1^A\gamma_1^B)^2\gamma_2^A\gamma_2^B\sqrt{\rho_{23}\rho_{32}\rho_{22}\rho_{33}}$. Again, $\beta'' \geq 0$ and the equality holds iff $t = \infty$ or $\rho_{23} = \rho_{32} = 0$. Thus ρ_I is also ESD-free under composite noise environment.

B. All states in ρ_{II}, ρ_{III} and ρ_{IV} disentangle abruptly

The situations in $\rho_{II\sim IV}$ are more complicated because the explicit expression of concurrence at finite time t is too lengthy to give a simple conclusion (due to the fact that the forms of $\rho_{II\sim IV}$ do not preserve under amplitude damping). Such complexity is also reflected in [15], where the derived necessary and sufficient condition for ESD under amplitude damping is mathematically packaged into the question of the positive definiteness for an artificial 4×4 matrix {see Eq.(11) of [15]}, which is still far from transparent.

However, using the factorization properties we can indeed prove that no state in $\rho_{II\sim IV}$ is ESD-free under composite noise environment. The trick is to divide the quantum channel $\$_t^C$ into two subsequent quantum channels $\$_\tau^C$ and $\$_{\tau'}^C$ as in Eq.(11). The evolution of a generic intial state $\rho(0)$ under $\$_t^C$ thus abides

$$\rho(t) = \$_t^C\rho(0) = \$_{\tau'}^C\$_\tau^C\rho(0) = \$_{\tau'}^C\rho(\tau), \tag{22}$$

where $\rho(\tau) = \$_\tau^C\rho(0)$. Therefore the final state $\rho(t)$ can be obtained by acting $\$_{\tau'}$ over the intermediate state $\rho(\tau)$. Now one asks what is the necessary and sufficient condition for asymptotic disentanglement for $\rho(0)$ under the action of $\$_t^C$? The answer is quite simple: *Neither has $\rho(\tau)$ completely disentangled within finite time τ, nor shall $\rho(\tau)$ disentangle completely under the action of $\$_{\tau'}^C$*, $\forall\,\tau, \tau' \geq 0$. The first part of the statement is no more than a repeat of the original question. However, the second part of the statement requires that over the full time evolution any intermediate state $\rho(\tau)$ should belong to the set of the ESD-free states under the action of $\$_{\tau'}^C$. Since the set of the ESD-free states under $\$_{\tau'}^C$ is contained in the subspaces $\rho_{I\sim IV}$, it is necessary for any intermediate state $\rho(\tau)$ to be located within $\rho_{I\sim IV}$ to guarantee asymptotic disentanglement. Now it is straightforward to show that as long as $\rho_{11} \neq 0$, such requirement cannot be satisfied whenever amplitude damping is present for each party: the diagonal elements of $\rho(\tau)$ are solely

determined by amplitude damping as

$$\begin{aligned}
\rho_{11}(\tau) &= (\gamma_1^A\gamma_1^B)^2\rho_{11}, \\
\rho_{22}(\tau) &= (\gamma_1^A)^2[\rho_{22}+(\omega_1^B)^2\rho_{11}], \\
\rho_{33}(\tau) &= (\gamma_1^B)^2[\rho_{33}+(\omega_1^A)^2\rho_{11}], \\
\rho_{44}(\tau) &= \rho_{44}+(\omega_1^B)^2\rho_{33}+(\omega_1^A)^2\rho_{22}+(\omega_1^A\omega_1^B)^2\rho_{11},
\end{aligned} \tag{23}$$

where $\gamma_1^{A,B}$, $\omega_1^{A,B}$ are functions of τ instead of t. It is obvious that if $\rho_{11}\neq 0$ and $\Gamma_1^{A,B}\neq 0$, none of the above diagonal elements vanishes $\forall\,\tau>0$. Therefore, we arrive at the conclusion that no state in $\rho_{II\sim IV}$ is ESD-free due to the existence of nonzero element ρ_{11} (Note that there exist states with $\rho_{11}=0$ in $\rho_{II\sim IV}$, however, this trivial ambiguity can be excluded by absorbing all these states into ρ_I).

V. DISCUSSION AND CONCLUSION

An interesting topic connected with our work is the violation of additivity for entanglement decay rates under weak noises discovered by Yu and Eberly [5]. They have demonstrated that a state disentangles asymptotically under either amplitude damping or phase damping may, however, disentangles abruptly under the combined action of both. This fact unequivocally manifests the violation of additivity for the decay rates of nonlocal quantum coherence, in sharp contrast to the uphold of that for local quantum coherences. In view of the ESD-free subspaces discussed, such phenomenon deserves a simple explanation. The specific example used in [5] is no more than a special state located in the subspace ρ_{IV}. Although it is ESD-free under either one of the noises, it is not ESD-free under both because the evolution of the state (due to amplitude damping) is not confined in the ESD-free subspace required by the composite noise environment. According to the previous analysis we conclude that such an abrupt violation of additivity can only occur for states located in the subspaces $\rho_{II\sim IV}$. On the other hand, due to the existence of the completely ESD-free subspace ρ_I, it is still possible to explore some quasi-additivity of entanglement decay rates for states within this restricted space.

Another important work to note is the finite temperature effect recently considered by Al-Qasimi and James [9] for amplitude damping (i.e., spontaneous excitation from $|\downarrow\rangle$ to $|\uparrow\rangle$ is also allowed due to thermal excitations in the reservoir). As demonstrated in an X-state example, they argue that all states disentangle asymptotically in zero-temperature bath must undergo sudden death at finite temperature. This is a very strong statement that may smear off the necessity to discriminate between the two kinds of disentanglement behaviors under realistic circumstances. Therefore, the significance of our work is restricted in the zero-temperature regime, where spontaneous excitation is forbidden.

In conclusion, we have shown that the set of ESD-free states for a two-qubit system under the combined action of amplitude damping and phase damping is fully represented by the subspace ρ_I, which is spanned on the bases $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. Therefore, any contamination from the double-excitation state $|\uparrow\uparrow\rangle$ in the initial density matrix is extremely hazardous for entanglement preservation. Entanglement resources realized via $|\uparrow\uparrow\rangle$ (e.g., the Bell-state $\phi_\pm$) thus should be treated with great caution comparing to those without it (e.g., the Bell-states $\psi_\pm$) whenever the lifetime of entanglement has to be taken into consideration.

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